

A duality approach to the symmetry of Bernstein-Sato polynomials of free divisors

L. Narváez Macarro*

Departamento de Álgebra & Instituto de Matemáticas (IMUS)
University of Sevilla

March 2012

Abstract

In this paper we prove that the Bernstein-Sato polynomial of any free divisor for which the $\mathcal{D}[s]$ -module $\mathcal{D}[s]h^s$ admits a Spencer logarithmic resolution satisfies the symmetry property $b(-s-2) = \pm b(s)$. This applies in particular to locally quasi-homogeneous free divisors, or more generally, to free divisors of linear Jacobian type. We also prove that the Bernstein-Sato polynomial of an integrable logarithmic connection \mathcal{E} and of its dual \mathcal{E}^* with respect to a free divisor of linear Jacobian type are related by the equality $b_{\mathcal{E}}(s) = \pm b_{\mathcal{E}^*}(-s-2)$. Our results are based on the behaviour of the modules $\mathcal{D}[s]h^s$ and $\mathcal{D}[s]\mathcal{E}[s]h^s$ under duality.

Keywords: Bernstein-Sato polynomials, free divisors, logarithmic differential operators, Spencer resolutions, Lie-Rinehart algebras, logarithmic connections.

MSC: 14F10, 32C38

Introduction

In [14] Granger and Schulze proved that the Bernstein-Sato polynomial of any reductive prehomogeneous determinant or of any regular special linear free divisor satisfies the equality $b(-s-2) = \pm b(s)$. Their proof is based on Sato's fundamental theorem for irreducible reductive prehomogeneous spaces. This symmetry property has been also checked for many other examples of linear (see for instance *loc. cit.* and [22]) and non-linear free divisors (e.g. quasi-homogeneous plane curves and the examples in [18]). In this paper we prove the above symmetry property for free divisors for which the $\mathcal{D}[s]$ -module $\mathcal{D}[s]h^s$ admits a logarithmic Spencer resolution (See Theorem (4.1) for a precise statement). This hypothesis holds for any free divisor of linear Jacobian type, and so for any locally quasi-homogeneous free divisor (for instance, free hyperplane arrangements or discriminants of stable maps in Mather's "nice dimensions"). The main ingredient of the proof is the explicit description of the $\mathcal{D}[s]$ -dual of $\mathcal{D}[s]h^s$ by means of the logarithmic duality formula in [5, 6].

*Partially supported by MTM2010-19298 and FEDER.

Let us mention that Yano proved in [27] that for any quasi-homogeneous germ $h : (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}, 0)$ with isolated singularity, its reduced Bernstein-Sato polynomial $\tilde{b}(s) = \frac{b(s)}{s+1}$ satisfies the equality $\tilde{b}(s) = \pm \tilde{b}(-s-d)$. Yano's result and ours suggest that both are extremal cases of a whole family of "pure" cases where symmetry properties occur with other intermediate shiftings (see Question (4.8)). One can expect even that in the "non-pure" cases, the factors of the Bernstein-Sato polynomial which break the symmetry appear as minimal polynomials of the action of s on other $\mathcal{D}[s]$ -modules attached to our singularity (see for instance the examples in [19, §3]), possibly related with the microlocal structure.

Let us now comment on the content of the paper.

In section §1 we recall the different conditions and hypotheses on free divisors we will use throughout the paper. In section §2 we recall the logarithmic Bernstein construction and we study the hypotheses we will need later to prove our main results. In section §3 we recall the duality formula in [6] and we apply it to describe the $\mathcal{D}[s]$ -dual of $\mathcal{D}[s]h^{\varphi(s)}$, where φ is a \mathbb{C} -algebra automorphism of $\mathbb{C}[s]$, under the hypotheses studied in section §2. In section §4 we prove the symmetry property $b(-s-2) = \pm b(s)$ under the above hypotheses. The idea of the proof is the following: once we know that the $\mathcal{D}[s]$ -dual of $\mathcal{D}[s]h^s$ (resp. of $\mathcal{D}[s]h^{s+1}$) is concentrated in degree 0 and is isomorphic to $\mathcal{D}[s]h^{-s-1}$ (resp. to $\mathcal{D}[s]h^{-s-2}$), we can compute the $\mathcal{D}[s]$ -dual of the exact sequence

$$0 \rightarrow \mathcal{D}[s]h^{s+1} \rightarrow \mathcal{D}[s]h^s \rightarrow \mathcal{Q} := (\mathcal{D}[s]h^s) / (\mathcal{D}[s]h^{s+1}) \rightarrow 0$$

and deduce that the $\mathcal{D}[s]$ -dual of \mathcal{Q} is concentrated in degree 1 and is isomorphic to $\mathcal{D}[s]h^{-s-2}/\mathcal{D}[s]h^{-s-1}$. From here the symmetry property comes up. At the end of the section we give some applications to the logarithmic comparison problem and a characterization of the logarithmic comparison theorem for Koszul free divisors. In section §5 we generalize the above results to the case of integrable logarithmic connections with respect to free divisors of linear Jacobian type.

I would like to thank Francisco Castro, Michel Granger and Mathias Schulze for useful discussions and comments.

1 Notations and linearity conditions

In this paper X will denote a complex manifold of pure dimension d , $D \subset X$ a hypersurface (= divisor), $\mathcal{O}_X[\star D]$ the sheaf of meromorphic functions along D , $\mathcal{O}_X(D)$ the sheaf of meromorphic functions along D with poles of order ≤ 1 and \mathcal{D}_X the sheaf of linear differential operators with coefficients in \mathcal{O}_X .

We will also denote by $\mathcal{I}_D \subset \mathcal{O}_X$ the Jacobian ideal of $D \subset X$, i.e. the coherent ideal of \mathcal{O}_X whose stalk at any $p \in X$ is the ideal generated by $h, h'_{x_1}, \dots, h'_{x_d}$, where $h \in \mathcal{O}_{X,p}$ is any reduced local equation of D at p and $x_1, \dots, x_d \in \mathcal{O}_{X,p}$ is a system of local coordinates centered at p .

We recall that D is a *free divisor*, in the sense of K. Saito [21], if the coherent \mathcal{O}_X -module $\mathcal{D}er_{\mathbb{C}}(\log D)$ of logarithmic vector fields with respect to D is locally free (of rank d). In such a case we will denote by $\mathcal{V}_X = \mathcal{O}_X[\mathcal{D}er_{\mathbb{C}}(\log D)] \subset \mathcal{D}_X$ the sheaf of logarithmic differential operators with respect to D [2].

(1.1) DEFINITION. (Cf. [26, §7.2]) Let A be a commutative ring and $I \subset A$ an ideal. We say that I is of linear type if the canonical (surjective) map of graded A -algebras $\text{Sym}_A(I) \rightarrow \text{Rees}(I)$ is an isomorphism.

In the above definition, if $I = (a_1, \dots, a_r)$ and (s_{i1}, \dots, s_{ir}) , $i \in L$, is a system of generators of the syzygies of a_1, \dots, a_r , to say that the ideal I is of linear type is equivalent to saying that any homogeneous polynomial $F(\xi_1, \dots, \xi_r) \in A[\xi]$ such that $F(a_1, \dots, a_r) = 0$ is a linear combination with coefficients in $A[\xi]$ of the linear forms $s_{i1}\xi_1 + \dots + s_{ir}\xi_r$, $i \in L$.

(1.2) EXAMPLE. An ideal generated by a regular sequence is of linear type.

(1.3) DEFINITION. ([7, Definitions 1.11, 1.14]) (a) We say that the divisor D is of linear Jacobian type at $p \in D$ if $\mathcal{J}_{D,p} \subset \mathcal{O}_{X,p}$ is of linear type. We say that D is of linear Jacobian type if it is so at any $p \in D$.

(b) We say that the divisor D is of differential linear type at $p \in D$ if for some (and hence for any) reduced local equation $h \in \mathcal{O}_{X,p}$ of D at p , the ideal $\text{ann}_{\mathcal{O}_{X,p}[s]} h^s$ is generated by order 1 operators. We say that D is of differential linear type if it is so at any $p \in D$.

Any divisor of linear Jacobian type is of differential linear type (cf. [7, Proposition 1.15]).

(1.4) DEFINITION. (a) We say that the divisor D is strongly Euler homogeneous if for any $p \in D$ and for some (and hence for any) reduced local equation $h \in \mathcal{O}_{X,p}$ of D at p there is a germ of vector field χ at p vanishing at p such that $\chi(h) = h$.

(b) We say that the divisor D is locally quasi-homogeneous if for any $p \in D$ there is a system of local coordinates $x = (x_1, \dots, x_d)$ centered at p such that the germ (D, p) has a reduced weighted homogeneous defining equation (with strictly positive weights) with respect to x .

(1.5) REMARK. There is also the notion of Euler homogeneity. Namely, we say that the divisor D is Euler homogeneous at a point $p \in D$ if there is a reduced local equation $h \in \mathcal{O}_{X,p}$ of D at p and a germ of vector field χ at p (not necessarily vanishing at p) such that $\chi(h) = h$. In that case we also say that h is Euler homogeneous. It is clear that if D is Euler homogeneous at p , then it is also Euler homogeneous at any point $q \in D$ close enough to p . Notice that not any local reduced equation of a Euler homogeneous divisor is Euler homogeneous. Notice also that for any divisor $D \subset X$, which could not be Euler homogeneous, the divisor $D' = D \times \mathbb{C} \subset X' = X \times \mathbb{C}$ is always Euler homogeneous. Nevertheless, a divisor $D \subset X$ is strongly Euler homogeneous if and only if $D' = D \times \mathbb{C} \subset X' = X \times \mathbb{C}$ is strongly Euler homogeneous. Let us also notice that a divisor $D \subset X$ is of linear Jacobian type at $p \in D$ if and only if $D' = D \times \mathbb{C} \subset X' = X \times \mathbb{C}$ is of linear Jacobian type at $(p, 0) \in D'$.

(1.6) EXAMPLE. Any smooth hypersurface is of linear Jacobian type. More generally, any quasi-homogeneous (with strictly positive weights) isolated singularity is of linear Jacobian type.

(1.7) PROPOSITION. If the divisor D is of linear Jacobian type, then it is strongly Euler homogeneous.

PROOF. Let $h \in \mathcal{O}_{X,p}$ be a reduced local equation of (D, p) and $x = (x_1, \dots, x_d)$ a system of local coordinates centered at p . By using that h belongs to the integral closure of the ideal $I = (h'_{x_1}, \dots, h'_{x_d})$ (cf. [23, §0.5, 1]) we deduce that in fact $h \in I$, i.e. that D is Euler homogeneous (see [7, Remark 1.26 (a)] for the details). Strong Euler homogeneity comes from the aforementioned facts that for a divisor $D \subset X$, D is strongly Euler homogeneous (resp. of linear Jacobian type) at a neighborhood of $p \in D$ if and only if $D' = D \times \mathbb{C} \subset X \times \mathbb{C}$ is strongly Euler homogeneous (resp. of linear Jacobian type) at a neighborhood of $(p, 0) \in D'$. Q.E.D.

It has been proven in [4, Theorem 5.6] that any locally quasi-homogeneous free divisor is of linear Jacobian type. We do not know any example of a free divisor of linear Jacobian type which is not locally quasi-homogeneous.

Let us recall that a free divisor D is said to be *Koszul* ([2, Definition 4.1.1]) at $p \in D$ if for some (and hence any) basis $\delta_1, \dots, \delta_d$ of $\mathcal{D}er_{\mathbb{C}}(\log D)_p$, the sequence $\sigma(\delta_1), \dots, \sigma(\delta_d)$ is regular in $\text{gr } \mathcal{D}_{X,p}[s]$. It turns out that this property is equivalent to being holonomic in the sense of Saito ([13, Theorem 7.4]).

The following definition is inspired by [14, Definition 7.1], which only applies to the case of linear free divisors (see also Proposition 7.1 and the subsequent remark in *loc. cit.*).

(1.8) DEFINITION. Assume that D is a free divisor. We say that D is strongly Koszul at $p \in D$ if for some (and hence any) basis $\delta_1, \dots, \delta_d$ of $\mathcal{D}er_{\mathbb{C}}(\log D)_p$ and for some (and hence any) reduced equation $h \in \mathcal{O}_{X,p}$ of (D, p) , the sequence

$$h, \sigma(\delta_1) - \alpha_1 s, \dots, \sigma(\delta_d) - \alpha_d s, \quad \text{with } \delta_i(h) = \alpha_i h,$$

is regular¹ in $\text{gr } \mathcal{D}_{X,p}[s]$.

(1.9) PROPOSITION. Assume that D is a free divisor and $p \in D$. The following properties are equivalent:

- (a) D is of linear Jacobian type at p .
- (b) D is strongly Koszul at p .

PROOF. Let $x_1, \dots, x_d \in \mathcal{O} := \mathcal{O}_{X,p}$ be a system of local coordinates centered at p , $h \in \mathcal{O}$ a reduced local equation of D at p and $J = \mathcal{J}_{D,p} = (h, h'_{x_1}, \dots, h'_{x_d})$. Let $\{\delta_i = \sum_{j=1}^d a_{ij} \frac{\partial}{\partial x_j}\}_{1 \leq i \leq d}$ be a basis of $\mathcal{D}er(\log D)_p$, and let us write $\delta_i(h) = \alpha_i h$ and $\sigma_i := \sigma(\delta_i) = \sum_{j=1}^d a_{ij} \xi_j \in \text{gr } \mathcal{D}_{X,p} = \mathcal{O}[\xi]$. The family $\{(-\alpha_i, a_{i1}, \dots, a_{id})\}_{1 \leq i \leq d}$ is a basis of the syzygies of $h, h'_{x_1}, \dots, h'_{x_d}$.

(a) \Rightarrow (b): From Proposition (1.7) we know that h is Euler homogeneous, i.e. $h \in (h'_{x_1}, \dots, h'_{x_d})$, and we can take $\alpha_1 = \dots = \alpha_{d-1} = 0$ and $\alpha_d = 1$. In other words, $\{(a_{i1}, \dots, a_{id})\}_{1 \leq i \leq d-1}$ is a basis of the syzygies of $h'_{x_1}, \dots, h'_{x_d}$.

Let $\varphi : \mathcal{O}[\xi] \rightarrow \text{Rees}(J) = \mathcal{O}[h'_{x_1} t, \dots, h'_{x_d} t]$ be the surjective map of \mathcal{O} -algebras defined by $\varphi(\xi_i) = h'_{x_i} t$. Since J is an ideal of linear type, the kernel of φ is generated by the σ_i , $1 \leq i \leq d-1$. So

$$\dim \left(\frac{\mathcal{O}[\xi]}{(\sigma_1, \dots, \sigma_{d-1})} \right) = \dim \text{Rees}(J) = d + 1$$

¹Since the sequence is formed by homogeneous elements in a graded ring, its regularity does not depend on the order.

and $\sigma_1, \dots, \sigma_{d-1}$ is a regular sequence in $\mathcal{O}[\xi]$.

On the other hand, since $\ker \varphi = (\sigma_1, \dots, \sigma_{d-1})$ is a prime ideal and $h \notin \ker \varphi$, we deduce that $h, \sigma_1, \dots, \sigma_{d-1}$ is also a regular sequence in $\mathcal{O}[\xi]$, or still $h, \sigma_1, \dots, \sigma_{d-1}, \sigma_d - s$ is a regular sequence in $\mathcal{O}[\xi, s] = \text{gr } \mathcal{D}_{X,p}[s]$ and D is strongly Koszul at p .

(b) \Rightarrow (a): Assume that X is a small enough open neighborhood of p , $\mathcal{K}^{(1)} = (\sigma_1 - \alpha_1 s, \dots, \sigma_d - \alpha_d s) \subset \mathcal{O}_X[s, \xi_1, \dots, \xi_d]$ and let \mathcal{K} be the kernel of the canonical graded surjective map

$$\Phi : \mathcal{O}_X[s, \xi_1, \dots, \xi_d] \rightarrow \text{Rees}(\mathcal{J}_D), \quad s \mapsto ht, \quad \xi_i \mapsto h'_{x_i} t. \quad (1)$$

Since D is of linear Jacobian type at any smooth point, we deduce that $\mathcal{K}/\mathcal{K}^{(1)}$ is supported by the singular locus of D . In particular, for any homogeneous polynomial $F \in \mathcal{K}_p$ there is a $N > 0$ such that $h^N F \in \mathcal{K}_p^{(1)}$, but $h, \sigma_1 - \alpha_1 s, \dots, \sigma_d - \alpha_d s$ is a regular sequence and so $F \in \mathcal{K}_p^{(1)}$. We deduce that $\mathcal{K}_p = \mathcal{K}_p^{(1)}$ and D is of linear Jacobian type at p . Q.E.D.

(1.10) COROLLARY. *Assume that D is a free divisor and $p \in D$. The following properties are equivalent:*

- (a) *D is strongly Koszul at p .*
- (b) *For any reduced equation $h \in \mathcal{O}_{X,p}$ of (D, p) and any basis $\delta_1, \dots, \delta_d$ of $\mathcal{D}er_{\mathbb{C}}(\log D)_p$ with $\delta_1(h) = \dots = \delta_{d-1}(h) = 0$ and $\delta_d(h) = h$, the sequence $h, \sigma(\delta_1), \dots, \sigma(\delta_{d-1})$ is regular in $\text{gr } \mathcal{D}_{X,p}$.*
- (c) *There is a reduced equation $h \in \mathcal{O}_{X,p}$ of (D, p) and a basis $\delta_1, \dots, \delta_d$ of $\mathcal{D}er_{\mathbb{C}}(\log D)_p$ with $\delta_1(h) = \dots = \delta_{d-1}(h) = 0$ and $\delta_d(h) = h$ such that the sequence $h, \sigma(\delta_1), \dots, \sigma(\delta_{d-1})$ is regular in $\text{gr } \mathcal{D}_{X,p}$.*

PROOF. It is a straightforward consequence of Propositions (1.9) and (1.7). Q.E.D.

Let us notice that property (c) in the above corollary appeared as condition (c') in [24, Corollary 1.8].

We recall the following definition of [19, page 257].

(1.11) DEFINITION. *Assume that D is a free divisor. We say that D is weakly Koszul² at $p \in D$ if for some (and hence any) basis $\delta_1, \dots, \delta_d$ of $\mathcal{D}er_{\mathbb{C}}(\log D)_p$ and some (and hence any) reduced local equation $h \in \mathcal{O}_{X,p}$ of (D, p) , the sequence*

$$\sigma(\delta_1) - \alpha_1 s, \dots, \sigma(\delta_d) - \alpha_d s, \quad \text{with } \delta_i(h) = \alpha_i h,$$

is regular in $\text{gr } \mathcal{D}_{X,p}[s]$. We say that D is weakly Koszul if it is so at any $p \in D$.

For a free divisor, the following implications hold: strongly Koszul \Rightarrow Koszul \Rightarrow weakly Koszul. The example $x_1 x_2 (x_1 + x_2)(x_1 + x_3 x_2) = 0$ is a weakly Koszul free divisor which is not Koszul ([19, Example 3.1]) and any non-quasihomogeneous plane curve is a Koszul free divisor non strongly Koszul (Proposition 2.3.1 in *loc. cit.*).

²This notion was called “(GK)” in [19].

(1.12) REMARK. Let D be a free divisor and $h \in \mathcal{O}_{X,p}$ a reduced local equation of (D, p) . If there is a germ of vector field χ at p such that $\chi(h) = h$ (i.e. h is Euler homogeneous), then D is weakly Koszul at p if and only if for some (and hence any) basis $\delta_1, \dots, \delta_{d-1}$ of germs of vector fields vanishing on h , the sequence $\sigma(\delta_1), \dots, \sigma(\delta_{d-1})$ is regular in $\text{gr } \mathcal{D}_{X,p}$.

2 Logarithmic–meromorphic comparison for Bernstein modules

From now on we assume that $h : (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}, 0)$ is a reduced local equation of a germ of free divisor $(D, 0) \subset (\mathbb{C}^d, 0)$. Let us write for short $\mathcal{O} = \mathcal{O}_{\mathbb{C}^d, 0}$, $\mathcal{D} = \mathcal{D}_{\mathbb{C}^d, 0}$ and $\mathcal{V} = \mathcal{V}_{\mathbb{C}^d, 0} = \mathcal{O}[\mathcal{D}er_{\mathbb{C}}(\log D)_0] \subset \mathcal{D}$. We consider the *logarithmic Bernstein module* $\mathcal{O}[s]h^s$ ([7, §1.6]), which is a $\mathcal{V}[s]$ -submodule of the Bernstein $\mathcal{D}[s]$ -module $\mathcal{O}[s, h^{-1}]h^s$ [1]. Obviously $\mathcal{O}[s]h^s$ is generated by h^s over $\mathcal{V}[s]$ and $\text{ann}_{\mathcal{V}[s]} h^s$ is the left $\mathcal{V}[s]$ -ideal generated by the Lie-Rinehart algebra over $(\mathbb{C}, \mathcal{O})$ (cf. [20])

$$\Theta_{h,s} := \{\delta - \alpha s \mid \delta \in \mathcal{D}er(\log D)_0, \delta(h) = \alpha h\} \subset \mathcal{V}[s].$$

The following result generalizes [24, Proposition 4.4] to the non-Euler homogeneous case and completes Proposition (1.9).

(2.1) PROPOSITION. *With the above hypotheses, the following properties are equivalent:*

- (a) $(D, 0)$ is of differential linear type and weakly Koszul.
- (b) $(D, 0)$ is of linear Jacobian type (or equivalently strongly Koszul).

PROOF. (b) \Rightarrow (a): It is a consequence of [7, Proposition 1.15] and [7, Proposition 1.27].

(a) \Rightarrow (b): We follow Torrelli's argument. Let $\delta_1, \dots, \delta_d$ be a basis of $\mathcal{D}er_{\mathbb{C}}(\log D)_0$ with $\delta_i(h) = \alpha_i h$ and let us write $K = \text{ann}_{\mathcal{D}[s]} h^s$. It is clear that $\Theta_{h,s}$ is freely generated as \mathcal{O} -module by $\delta_1 - \alpha_1 s, \dots, \delta_d - \alpha_d s$. Since $(D, 0)$ is of differential linear type, we have $K = \mathcal{D}[s]\Theta_{h,s}$. Let us consider the filtration by the total order in $\mathcal{D}[s]$ (the total order of s is 1; cf. [27, Definition 2.1]). Since $\sigma(\delta_1) - \alpha_1 s, \dots, \sigma(\delta_d) - \alpha_d s$ is a regular sequence in $\text{gr } \mathcal{D}[s] = \text{gr}_T(\mathcal{D}[s])$, we deduce that $\sigma_T(K)$ is the ideal of $\text{gr}_T(\mathcal{D}[s])$ generated by $\sigma(\delta_1) - \alpha_1 s, \dots, \sigma(\delta_d) - \alpha_d s$. We know that the characteristic variety $\widetilde{W} = V(\sigma_T(K)) \subset \mathbb{C} \times T^*\mathbb{C}^d$ of $\mathcal{D}[s]h^s$ is irreducible of dimension $d + 1$ ([15, §5], [27, Proposition 2.3]). In fact $I(\widetilde{W}) = \ker \Phi$, where Φ has been defined in (1). Since $\Phi(h) \neq 0$ we deduce that $\dim V(h, \sigma(\delta_1) - \alpha_1 s, \dots, \sigma(\delta_d) - \alpha_d s) = \dim(W \cap V(h)) = d$ and so $h, \sigma(\delta_1) - \alpha_1 s, \dots, \sigma(\delta_d) - \alpha_d s$ is a regular sequence. Q.E.D.

Let us denote by $\text{Sp}_{\Theta_{h,s}}^\bullet$ the Cartan-Eilenberg-Chevalley-Rinehart-Spencer complex defined by (see 1.1.2 in *loc. cit.*) $\text{Sp}_{\Theta_{h,s}}^{-r} = \mathcal{V}[s] \otimes_{\mathcal{O}} \bigwedge^r \Theta_{h,s}$, $r \geq 0$, and

the differential $\varepsilon^{-r} : \mathrm{Sp}_{\Theta_{h,s}}^{-r} \rightarrow \mathrm{Sp}_{\Theta_{h,s}}^{-(r-1)}$ is given by:

$$\begin{aligned} \varepsilon^{-r}(P \otimes (\lambda_1 \wedge \cdots \wedge \lambda_r)) &= \\ &= \sum_{i=1}^r (-1)^{i-1} (P\lambda_i) \otimes (\lambda_1 \wedge \cdots \wedge \widehat{\lambda_i} \wedge \cdots \wedge \lambda_r) + \\ &+ \sum_{1 \leq i < j \leq r} (-1)^{i+j} P \otimes ([\lambda_i, \lambda_j] \wedge \lambda_1 \wedge \cdots \wedge \widehat{\lambda_i} \wedge \cdots \wedge \widehat{\lambda_j} \wedge \cdots \wedge \lambda_r) \end{aligned}$$

for $r \geq 2$, and $\varepsilon^{-1}(P \otimes \lambda_1) = P\lambda_1$ for $r = 1$, and $P \in \mathcal{V}[s]$, $\lambda_i \in \Theta_{h,s}$.

From Proposition 1.21 in *loc. cit.*, we know that $\mathrm{Sp}_{\Theta_{h,s}}^\bullet$ becomes a $\mathcal{V}[s]$ -free resolution of $\mathcal{O}[s]h^s$ with the augmentation $\varepsilon^0 : \mathrm{Sp}_{\Theta_{h,s}}^0 = \mathcal{V}[s] \rightarrow \mathcal{O}[s]h^s$, $\varepsilon^0(P) = Ph^s$.

The proof of the following proposition is clear.

(2.2) PROPOSITION. *Under the above hypotheses, the following properties are equivalent:*

- (a) *The canonical map $\mathcal{D}[s] \overset{\mathbf{L}}{\otimes}_{\mathcal{V}[s]} (\mathcal{O}[s]h^s) \rightarrow \mathcal{D}[s]h^s$ is an isomorphism in the derived category of left $\mathcal{D}[s]$ -modules.*
- (b) *The divisor D is of differential linear type at 0 and the complex $\mathcal{D}[s] \otimes_{\mathcal{V}[s]} \mathrm{Sp}_{\Theta_{h,s}}^\bullet$ is exact in degrees $\neq 0$.*

(2.3) PROPOSITION. *Any germ of free divisor $(D, 0) \subset (\mathbb{C}^d, 0)$ of differential linear type and weakly Koszul at 0 satisfies the equivalent properties of Proposition (2.2).*

PROOF. To prove that the complex $L = \mathcal{D}[s] \otimes_{\mathcal{V}[s]} \mathrm{Sp}_{\Theta_{h,s}}^\bullet$ is exact in degrees $\neq 0$, we filter L in such a way that its graded complex is the Koszul complex associated with the sequence $\sigma(\delta_1) - \alpha_1 s, \dots, \sigma(\delta_d) - \alpha_d s$ with $\delta_1, \dots, \delta_d$ a basis of $\mathrm{Der}_{\mathbb{C}}(\log D)_0$ and $\delta_i(h) = \alpha_i h$ (see the proof of [19, Corollary 2.2.18]). Q.E.D.

The following corollary is a particular case of [7, Theorem 3.1].

(2.4) COROLLARY. *Under the above hypotheses, if $(D, 0) \subset (\mathbb{C}^d, 0)$ is a germ of free divisor of linear Jacobian type, then the equivalent properties of Proposition (2.2) hold.*

PROOF. It is clear from Proposition (2.1).

Q.E.D.

(2.5) DEFINITION. *For any polynomial $q(s) \in \mathbb{C}[s]$ we define:*

- (1) *The $q(s)$ -Bernstein module as the free $\mathcal{O}[s, h^{-1}]$ -module $\mathcal{O}[s, h^{-1}]h^{q(s)}$ with basis $h^{q(s)}$ endowed with the left $\mathcal{D}[s]$ -module structure given by*

$$\delta \cdot (ah^{q(s)}) = (\delta(a) + q(s)\delta(h)h^{-1}a)h^{q(s)}$$

for any $\delta \in \mathrm{Der}_{\mathbb{C}}(\mathcal{O})$.

- (2) *The logarithmic $q(s)$ -Bernstein module as the left $\mathcal{V}[s]$ -submodule $\mathcal{O}[s]h^{q(s)}$ of the $q(s)$ -Bernstein module $\mathcal{O}[s, h^{-1}]h^{q(s)}$.*

It is clear that $\mathcal{O}[s]h^{q(s)}$ is generated by $h^{q(s)}$ over $\mathcal{V}[s]$ and $\text{ann}_{\mathcal{V}[s]} h^{q(s)}$ is the left $\mathcal{V}[s]$ -ideal generated by the $(\mathbb{C}, \mathcal{O})$ -Lie-Rinehart algebra

$$\Theta_{h,q(s)} := \{\delta - \alpha q(s) \mid \delta \in \mathcal{D}er(\log D)_0, \delta(h) = \alpha h\}.$$

For any \mathbb{C} -algebra map $\varphi : \mathbb{C}[s] \rightarrow \mathbb{C}[s]$ let us also call φ its trivial extensions to $\mathcal{O}[s]$, $\mathcal{O}[s, h^{-1}]$, $\mathcal{D}[s]$ and $\mathcal{V}[s]$. For any $q(s) \in \mathbb{C}[s]$ the map

$$\overline{\varphi} : ah^{q(s)} \in \mathcal{O}[s, h^{-1}]h^{q(s)} \mapsto \varphi(a)h^{\varphi(q(s))} \in \mathcal{O}[s, h^{-1}]h^{\varphi(q(s))}$$

$$(\text{resp. } \overline{\varphi} : ah^{q(s)} \in \mathcal{O}[s]h^{q(s)} \mapsto \varphi(a)h^{\varphi(q(s))} \in \mathcal{O}[s]h^{\varphi(q(s))})$$

is linear over $\varphi : \mathcal{D}[s] \rightarrow \mathcal{D}[s]$ (resp. over $\varphi : \mathcal{V}[s] \rightarrow \mathcal{V}[s]$).

Since $\varphi(\Theta_{h,q(s)}) = \Theta_{h,\varphi(q(s))}$, the natural map

$$\varphi^* \left(\mathcal{O}[s]h^{q(s)} \right) := \mathcal{V}[s] \otimes_{\varphi} \left(\mathcal{O}[s]h^{q(s)} \right) \rightarrow \mathcal{O}[s]h^{\varphi(q(s))}$$

induced by $\overline{\varphi} : \mathcal{O}[s]h^{q(s)} \rightarrow \mathcal{O}[s]h^{\varphi(q(s))}$, where $\mathcal{V}[s] \otimes_{\varphi} (-)$ denote the scalar extension associated with $\varphi : \mathcal{V}[s] \rightarrow \mathcal{V}[s]$, is an isomorphism of left $\mathcal{V}[s]$ -modules.

The proof of the following lemma is straightforward.

(2.6) LEMMA. *If $\varphi : \mathbb{C}[s] \rightarrow \mathbb{C}[s]$ is an automorphism, then the natural maps*

$$\varphi^* \left(\mathcal{O}[s, h^{-1}]h^{q(s)} \right) := \mathcal{D}[s] \otimes_{\varphi} \left(\mathcal{O}[s, h^{-1}]h^{q(s)} \right) \rightarrow \mathcal{O}[s, h^{-1}]h^{\varphi(q(s))}$$

and

$$\varphi^* \left(\mathcal{D}[s]h^{q(s)} \right) := \mathcal{D}[s] \otimes_{\varphi} \left(\mathcal{D}[s]h^{q(s)} \right) \rightarrow \mathcal{D}[s]h^{\varphi(q(s))}$$

induced by $\overline{\varphi} : \mathcal{O}[s, h^{-1}]h^{q(s)} \rightarrow \mathcal{O}[s, h^{-1}]h^{\varphi(q(s))}$, where $\mathcal{D}[s] \otimes_{\varphi} (-)$ denote the scalar extension associated with $\varphi : \mathcal{D}[s] \rightarrow \mathcal{D}[s]$, are isomorphisms of left $\mathcal{D}[s]$ -modules.

(2.7) PROPOSITION. *Assume that $\varphi : \mathbb{C}[s] \rightarrow \mathbb{C}[s]$ is an automorphism of \mathbb{C} -algebras. Then, the following properties are equivalent to the properties of Proposition (2.2):*

- (a') *The canonical map $\mathcal{D}[s] \overset{\mathbf{L}}{\otimes}_{\mathcal{V}[s]} (\mathcal{O}[s]h^{\varphi(s)}) \rightarrow \mathcal{D}[s]h^{\varphi(s)}$ is an isomorphism in the derived category of left $\mathcal{D}[s]$ -modules.*
- (b') *$\text{ann}_{\mathcal{D}[s]} h^{\varphi(s)}$ is the left $\mathcal{V}[s]$ -ideal generated by $\Theta_{h,\varphi(s)}$ and the complex $\mathcal{D}[s] \otimes_{\mathcal{V}[s]} \text{Sp}_{\Theta_{h,\varphi(s)}}^{\bullet}$ is exact in degrees $\neq 0$, where $\text{Sp}_{\Theta_{h,\varphi(s)}}^{\bullet}$ is defined in a completely similar way to $\text{Sp}_{\Theta_{h,s}}^{\bullet}$.*

PROOF. The proof is a straightforward consequence of the fact that for any left $\mathcal{V}[s]$ -module \mathcal{M} there is a canonical isomorphism of left $\mathcal{D}[s]$ -modules

$$\varphi^* \left(\mathcal{D}[s] \otimes_{\mathcal{V}[s]} \mathcal{M} \right) \simeq \mathcal{D}[s] \otimes_{\mathcal{V}[s]} \varphi^*(\mathcal{M}).$$

Q.E.D.

3 Duality

We keep the notations of §2. The free \mathcal{O} -modules $\mathcal{D}er_{\mathbb{C}}(\log D)_0$ and $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X)_0 = \mathcal{D}er_{\mathbb{C}}(\mathcal{O})$ are $(\mathbb{C}, \mathcal{O})$ -Lie-Rinehart algebras whose respective envelopping (or universal) algebras (cf. [20]) are \mathcal{V} and \mathcal{D} respectively. By the scalar extension $\mathbb{C} \rightarrow \mathbb{C}[s]$ we obtain the $(\mathbb{C}[s], \mathcal{O}[s])$ -Lie-Rinehart algebras $\mathcal{S} := \mathcal{D}er_{\mathbb{C}}(\log D)_0[s]$ and $\mathcal{S}' := \mathcal{D}er_{\mathbb{C}}(\mathcal{O})[s] = \mathcal{D}er_{\mathbb{C}[s]}(\mathcal{O}[s])$, and their envelopping algebras are $\mathcal{V}[s]$ and $\mathcal{D}[s]$ respectively. The rings $\mathcal{V}[s]$ and $\mathcal{D}[s]$ are left and right noetherian of finite global homological dimension.

Let us call $\omega := \bigwedge^d \Omega_{\mathcal{O}/\mathbb{C}}^1 = \Omega_{\mathbb{C}^d,0}^d$, $\omega(\log D) := \Omega_{\mathbb{C}^d,0}^d(\log D) = \bigwedge^d \Omega_{\mathbb{C}^d,0}^1(\log D)$, $\omega_{\mathcal{S}'} := \omega[s]$ and $\omega_{\mathcal{S}} := \omega(\log D)[s]$.

(3.1) PROPOSITION. (1) *The free $\mathcal{O}[s]$ -module of rank one $\omega_{\mathcal{S}'}$ (resp. $\omega_{\mathcal{S}}$) has a canonical right $\mathcal{D}[s]$ -module structure (resp. right $\mathcal{V}[s]$ -module structure).*

(2) *We have a canonical $\mathcal{V}[s]$ -linear isomorphism (resp. $\mathcal{D}[s]$ -linear isomorphism) $\text{Ext}_{\mathcal{V}[s]}^d(\mathcal{O}[s], \mathcal{V}[s]) \simeq \omega_{\mathcal{S}}$ (resp. $\text{Ext}_{\mathcal{D}[s]}^d(\mathcal{O}[s], \mathcal{D}[s]) \simeq \omega_{\mathcal{S}'}$), and for all $i \neq d$, $\text{Ext}_{\mathcal{V}[s]}^i(\mathcal{O}[s], \mathcal{V}[s]) = 0$ (resp. $\text{Ext}_{\mathcal{D}[s]}^i(\mathcal{O}[s], \mathcal{D}[s]) = 0$).*

PROOF. Part (1) is clear from the canonical right \mathcal{D} -module structure (resp. right \mathcal{V} -module structure) on ω (resp on $\omega(\log D)$). The proof of part (2) is identical to the proof of [5, Proposition 3.1]. In fact we can use

$$\text{Ext}_{\mathcal{A}[s]}^i(\mathcal{O}[s], \mathcal{A}[s]) = \text{Ext}_{\mathcal{A}}^i(\mathcal{O}_X, \mathcal{A}) \otimes_{\mathbb{C}} \mathbb{C}[s], \quad \text{for } \mathcal{A} = \mathcal{V} \text{ or } \mathcal{D}.$$

Q.E.D.

As in [5, §2], for any left $\mathcal{D}[s]$ -module (resp. left $\mathcal{V}[s]$ -module) \mathcal{M} , the $\mathcal{O}[s]$ -module $\omega_{\mathcal{S}'} \otimes_{\mathcal{O}[s]} \mathcal{M}$ (resp. $\omega_{\mathcal{S}} \otimes_{\mathcal{O}[s]} \mathcal{M}$) has a canonical right $\mathcal{D}[s]$ -module structure (resp. right $\mathcal{V}[s]$ -module structure), and it will be denoted by $\mathcal{M}^{\text{right}}$. Similarly, for any right $\mathcal{D}[s]$ -module (resp. right $\mathcal{V}[s]$ -module) \mathcal{N} , the $\mathcal{O}[s]$ -module $\text{Hom}_{\mathcal{O}[s]}(\omega_{\mathcal{S}'}, \mathcal{N})$ (resp. $\text{Hom}_{\mathcal{O}[s]}(\omega_{\mathcal{S}}, \mathcal{N})$) has a canonical left $\mathcal{D}[s]$ -module structure (resp. left $\mathcal{V}[s]$ -module structure) and it will be denoted by $\mathcal{N}^{\text{left}}$. Moreover we have canonical $\mathcal{D}[s]$ -linear isomorphisms (resp. $\mathcal{V}[s]$ -linear isomorphisms)

$$\mathcal{M} \simeq (\mathcal{M}^{\text{right}})^{\text{left}}, \quad \mathcal{N} \simeq (\mathcal{N}^{\text{left}})^{\text{right}}.$$

In a similar way, for any left $\mathcal{D}[s]$ -modules (resp. left $\mathcal{V}[s]$ -modules) \mathcal{M} and \mathcal{M}' , the $\mathcal{O}[s]$ -modules $\mathcal{M} \otimes_{\mathcal{O}[s]} \mathcal{M}'$ and $\text{Hom}_{\mathcal{O}[s]}(\mathcal{M}, \mathcal{M}')$ have a canonical left $\mathcal{D}[s]$ -module structure (resp. left $\mathcal{V}[s]$ -module structure). If \mathcal{M} is locally free of finite rank over $\mathcal{O}[s]$ the left $\mathcal{D}[s]$ -module (resp. left $\mathcal{V}[s]$ -module) $\text{Hom}_{\mathcal{O}[s]}(\mathcal{M}, \mathcal{O}[s])$ is denoted by \mathcal{M}^* .

(3.2) DEFINITION. *Let $D_f^b(\mathcal{V}[s])$ and $D_f^b(\mathcal{D}[s])$ be respectively the bounded derived categories of left $\mathcal{V}[s]$ -modules and of left $\mathcal{D}[s]$ -modules with finitely generated cohomologies. The duality functors $\mathbb{V} : D_f^b(\mathcal{V}[s]) \rightarrow D_f^b(\mathcal{V}[s])$ and $\mathbb{D} : D_f^b(\mathcal{D}[s]) \rightarrow D_f^b(\mathcal{D}[s])$ are defined by*

$$\mathbb{V}(\mathcal{M}) = (R\text{Hom}_{\mathcal{V}[s]}(\mathcal{M}, \mathcal{V}[s])[d])^{\text{left}}, \quad \mathbb{D}(\mathcal{M}) = (R\text{Hom}_{\mathcal{D}[s]}(\mathcal{M}, \mathcal{D}[s])[d])^{\text{left}}.$$

The above functors are contravariant involutive triangulated functors. Let us notice that we have canonical isomorphisms $\mathbb{V}(\mathcal{O}[s]) \simeq \mathcal{O}[s]$, $\mathbb{D}(\mathcal{O}[s]) \simeq \mathcal{O}[s]$.

(3.3) PROPOSITION. *Let \mathcal{M} be a left $\mathcal{V}[s]$ -module (resp. a left $\mathcal{D}[s]$ -module). If \mathcal{M} is locally free of finite rank over $\mathcal{O}[s]$, we have a canonical $\mathcal{V}[s]$ -isomorphism (resp. $\mathcal{D}[s]$ -isomorphism) $\mathbb{V}(\mathcal{M}) \simeq \mathcal{M}^*$ (resp. $\mathbb{D}(\mathcal{M}) \simeq \mathcal{M}^*$).*

PROOF. It is identical to the proof of [5, Proposition 3.1]. Q.E.D.

Let us write $\mathcal{O}(D)$ for the stalk at the origin of the integrable logarithmic connection $\mathcal{O}_{\mathbb{C}^d}(D)$ (see [5, §1.2]). The following theorem is a particular case of [6, Theorem 4.5].

(3.4) THEOREM. *For any complex \mathcal{M} in $D_f^b(\mathcal{V}[s])$ we have a canonical isomorphism in $D_c^b(\mathcal{D}[s])$*

$$\mathbb{D} \left(\mathcal{D}[s] \overset{\mathbf{L}}{\otimes}_{\mathcal{V}[s]} \mathcal{M} \right) \simeq \mathcal{D}[s] \overset{\mathbf{L}}{\otimes}_{\mathcal{V}[s]} \mathbb{V}(\mathcal{M})(D).$$

where $\mathbb{V}(\mathcal{M})(D) = \mathbb{V}(\mathcal{M}) \otimes_{\mathcal{O}[s]} \mathcal{O}(D)[s]$.

(3.5) REMARK. Let us notice that [6, Theorem 4.5] is a straightforward generalization of [5, Corollary 3.1.2], and the existence of a such result has been suggested by [12, Appendix A, Proposition (A.2)], [25] and [9, Theorem 4.3]. A related result is [11, Lemma 4.3.3].

(3.6) COROLLARY. *For any polynomial $q(s) \in \mathbb{C}[s]$, there is a canonical isomorphism*

$$\mathbb{D} \left(\mathcal{D}[s] \overset{\mathbf{L}}{\otimes}_{\mathcal{V}[s]} \mathcal{O}[s] h^{q(s)} \right) \simeq \mathcal{D}[s] \overset{\mathbf{L}}{\otimes}_{\mathcal{V}[s]} \mathcal{O}[s] h^{-q(s)-1}.$$

PROOF. It is a consequence of of canonical isomorphisms $(\mathcal{O}[s] h^{q(s)})^* \simeq \mathcal{O}[s] h^{-q(s)}$ and $(\mathcal{O}[s] h^{-q(s)})(D) \simeq \mathcal{O}[s] h^{-q(s)-1}$. Q.E.D.

(3.7) COROLLARY. *Under the above hypotheses, assume that our germ $h : (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}, 0)$ satisfies the equivalent properties of Proposition (2.2) and let $\varphi : \mathbb{C}[s] \rightarrow \mathbb{C}[s]$ an automorphism of \mathbb{C} -algebras. Then, there is a canonical isomorphism*

$$\mathbb{D} \left(\mathcal{D}[s] h^{\varphi(s)} \right) \simeq \mathcal{D}[s] h^{-\varphi(s)-1}.$$

PROOF. It is a consequence of the above corollary and Proposition (2.7). Q.E.D.

4 The symmetry of Bernstein-Sato polynomials

In this section we keep the notations of §3.

(4.1) THEOREM. *Let $h : (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}, 0)$ be a non-constant reduced germ of holomorphic function such that the divisor $D = h^{-1}(0)$ is free and satisfies the equivalent properties of Proposition (2.2). Then its Bernstein-Sato polynomial satisfies the equality $b(s) = \pm b(-s-2)$.*

PROOF. Let us consider the exact sequence of left $\mathcal{D}[s]$ -modules

$$0 \rightarrow \mathcal{D}[s]h^{s+1} \rightarrow \mathcal{D}[s]h^s \rightarrow \mathcal{Q} := (\mathcal{D}[s]h^s) / (\mathcal{D}[s]h^{s+1}) \rightarrow 0.$$

The Bernstein-Sato polynomial $b(s)$ of h is by definition the minimal polynomial of the action of s on \mathcal{Q} . By applying the duality functor \mathbb{D} we obtain a triangle

$$\mathbb{D}(\mathcal{Q}) \rightarrow \mathbb{D}(\mathcal{D}[s]h^s) \rightarrow \mathbb{D}(\mathcal{D}[s]h^{s+1}) \xrightarrow{+1}$$

and from corollary (3.7) we deduce that the second arrow corresponds to the inclusion $\mathcal{D}[s]h^{-s-1} \rightarrow \mathcal{D}[s]h^{-s-2}$, $\mathbb{D}(\mathcal{Q})$ is concentrated in degree 1 and there is an exact sequence of left $\mathcal{D}[s]$ -modules

$$0 \rightarrow \mathcal{D}[s]h^{-s-1} \rightarrow \mathcal{D}[s]h^{-s-2} \rightarrow \mathbb{D}^1(\mathcal{Q}) \rightarrow 0.$$

Let us call $\varphi : \mathbb{C}[s] \rightarrow \mathbb{C}[s]$ the automorphism of \mathbb{C} -algebras determined by $\varphi(s) = -s - 2$. From Lemma (2.6) we deduce that $\varphi^*(\mathcal{Q}) \simeq \mathbb{D}^1(\mathcal{Q})$ and so the minimal polynomial of the action of s on $\mathbb{D}^1(\mathcal{Q})$ is $\varphi(b(s)) = b(-s - 2)$. On the other hand, since the action of s on \mathcal{Q} is annihilated by $b(s)$ we deduce that the action of s on $\mathbb{D}^1(\mathcal{Q})$ is also annihilated by $b(s)$ and we conclude that $b(s)$ is a multiple of $b(-s - 2)$, i.e. $b(s) = \pm b(-s - 2)$. Q.E.D.

(4.2) COROLLARY. *Let $(D, 0) \subset (\mathbb{C}^d, 0)$ a germ of free divisor of linear Jacobian type. Then its Bernstein-Sato polynomial satisfies the equality $b(s) = \pm b(-s - 2)$.*

(4.3) COROLLARY. *Under the hypotheses of Theorem (4.1) the Bernstein-Sato polynomial of h has no (integer) roots less or equal than -2 .*

(4.4) REMARK. Let us notice that the above corollary applies in particular to the case of (weakly) Koszul free divisors of differential linear type (or equivalently, to free divisors of linear Jacobian type; see Proposition (2.1)) and so it answers (partially) the question stated in [24, Remark 4.7].

(4.5) QUESTION. We do not know whether reductive prehomogeneous determinants or regular special linear free divisors satisfy the equivalent properties of Proposition (2.2) or not, and so we do not know whether the results by Granger and Schulze in [14] can be derived from our Theorem (4.1) or not. However, all the examples of free divisors given in [18] are of linear Jacobian type and so they satisfy the above properties.

(4.6) COROLLARY. *Let $D \subset X$ be a free divisor and assume that the equivalent properties of Proposition (2.2) hold for some (and hence any) local reduced equation of D at each point $p \in D$. Then, the canonical map $\mathcal{D}_X \otimes_{\mathcal{V}_X}^{\mathbf{L}} \mathcal{O}_X(D) \rightarrow \mathcal{O}_X[\star D]$ is an isomorphism and the logarithmic comparison theorem holds.*

PROOF. The problem being local, we can assume that $p = 0 \in \mathbb{C}^d$ and $(D, 0) \subset (\mathbb{C}^d, 0)$ is given by a reduced equation $h \in \mathcal{O}$. Since -1 is the smallest integer root of the Bernstein-Sato polynomial of h , we deduce that the \mathcal{D} -module $\mathcal{O}[\star D]$ is generated by h^{-1} and that $\text{ann}_{\mathcal{D}} h^{-1}$ is obtained from $\text{ann}_{\mathcal{D}[s]} h^s$ by specializing $s = -1$, and so $\text{ann}_{\mathcal{D}} h^{-1}$ is generated by order 1 differential operators. In

other words, the canonical map $\mathcal{D} \otimes_{\mathcal{V}} \mathcal{O}(D) \rightarrow \mathcal{O}[\star D]$ is an isomorphism of left \mathcal{D} -modules.

In order to prove that the complex $\mathcal{D} \otimes_{\mathcal{V}}^{\mathbf{L}} \mathcal{O}(D)$ is concentrated in degree 0, we proceed as in [19, Proposition 2.2.17]. First, since $\mathcal{O}[s]h^s$ has no $(s+1)$ -torsion, we have

$$\mathcal{O}(D) = \mathcal{O}h^{-1} \simeq (\mathcal{V}[s]/\mathcal{V}[s](s+1)) \otimes_{\mathcal{V}[s]} \mathcal{O}[s]h^s \simeq (\mathcal{V}[s]/\mathcal{V}[s](s+1)) \otimes_{\mathcal{V}[s]}^{\mathbf{L}} \mathcal{O}[s]h^s,$$

and second

$$\begin{aligned} \mathcal{D} \otimes_{\mathcal{V}}^{\mathbf{L}} \mathcal{O}(D) &\simeq \cdots \simeq \mathcal{D} \otimes_{\mathcal{V}}^{\mathbf{L}} (\mathcal{V}[s]/\mathcal{V}[s](s+1)) \otimes_{\mathcal{V}[s]}^{\mathbf{L}} \mathcal{O}[s]h^s \simeq \\ &(\mathcal{D}[s]/\mathcal{D}[s](s+1)) \otimes_{\mathcal{V}[s]}^{\mathbf{L}} \mathcal{O}[s]h^s \simeq (\mathcal{D}[s]/\mathcal{D}[s](s+1)) \otimes_{\mathcal{D}[s]}^{\mathbf{L}} \mathcal{D}[s] \otimes_{\mathcal{V}[s]}^{\mathbf{L}} \mathcal{O}[s]h^s \simeq \\ &(\mathcal{D}[s]/\mathcal{D}[s](s+1)) \otimes_{\mathcal{D}[s]}^{\mathbf{L}} \mathcal{D}[s]h^s, \end{aligned}$$

but $\mathcal{D}[s]h^s \subset \mathcal{O}[s, h^{-1}]h^s$ has no $(s+1)$ -torsion and so the last complex is concentrated in degree 0.

The last statement is a consequence of [5, Theorem 4.1].

Q.E.D.

(4.7) REMARK. Let us notice that, after corollary (2.4), the above corollary applies to free divisors of linear Jacobian type, and so to locally quasi-homogeneous free divisors ([4, Theorem 5.6]). In particular we obtain a purely algebraic proof of the logarithmic comparison Theorem in [8] (see [7, Remark 1.25]).

(4.8) QUESTION. In [27, Corollary 3.9], Yano proved that for any quasi-homogeneous polynomial $h : \mathbb{C}^d \rightarrow \mathbb{C}$ with an isolated singularity at the origin, the reduced Bernstein-Sato polynomial $\tilde{b}(s) = \frac{b(s)}{s+1}$ satisfies the equality $\tilde{b}(s) = \pm \tilde{b}(-s-d)$. The intersection of Theorem (4.1) and Yano's result is the case of quasi-homogeneous plane curves. Quasi-homogeneous isolated singularities are of linear Jacobian type and their Jacobian ideal are a complete intersection, and so Cohen-Macaulay. On the other hand, the Jacobian ideal of a singular free divisor is also Cohen-Macaulay of codimension 2. The following natural question appears: let $(D, 0) \subset (\mathbb{C}^d, 0)$ be a singular germ of hypersurface of linear Jacobian type whose Jacobian ideal is Cohen-Macaulay of codimension e . Does its reduced Bernstein-Sato polynomial $\tilde{b}(s)$ satisfy the equality $\tilde{b}(s) = \pm \tilde{b}(-s-e)$?

The following result generalizes [14, Theorem 1.6] for any Koszul free divisor, not necessarily reductive linear, and [19, Proposition 2.3.1] for higher dimension. It also improves [24, Corollary 1.8].

(4.9) THEOREM. *Let $D \subset X$ be a Koszul free divisor. The following properties are equivalent:*

- (a) *D is strongly Euler homogeneous.*
- (b) *D is of linear Jacobian type.*
- (c) *D is strongly Koszul.*
- (d) *D is of differential linear type.*

(e) D satisfies de logarithmic comparison theorem.

PROOF. (a) \Rightarrow (b) We follow the lines of the proof of [4, Theorem 5.6] by induction on $\dim X$. For $\dim X = 2$ the result is known (cf. [19, Proposition 2.3.1]). Assume that the result is true whenever the ambient manifold has dimension $d - 1$ and assume now that $\dim X = d \geq 3$.

Let $p \in D$ be a point. The question being local, we can assume that $X \subset \mathbb{C}^d$ is a small open neighborhood of $p = 0 \in D$. Let $h : X \rightarrow \mathbb{C}$ be a reduced equation of D and $\mathcal{J}_D = (h, h'_{x_1}, \dots, h'_{x_d}) \subset \mathcal{O}_X$ its Jacobian. Let $\{\delta_i = \sum_{j=1}^d a_{ij} \frac{\partial}{\partial x_j}\}_{1 \leq i \leq d}$ be a basis of $\Gamma(X, \mathcal{D}er_{\mathbb{C}}(\log D))$. Since D is strongly Euler homogeneous we can take $\delta_i(h) = 0$ for all $i = 1, \dots, d - 1$, $\delta_d(h) = h$ and $a_{d1}(0) = \dots = a_{dd}(0) = 0$. In particular $h \in (h'_{x_1}, \dots, h'_{x_d})$ and $\mathcal{J}_D = (h'_{x_1}, \dots, h'_{x_d})$.

The kernel of the natural map $\mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X) \rightarrow \mathcal{J}_D$ sending any derivation δ to $\delta(h)$ coincides with the free \mathcal{O}_X -submodule $\tilde{\Theta}_h \subset \mathcal{D}er_{\mathbb{C}}(\log D)$ generated by $\delta_1, \dots, \delta_{d-1}$ and $\mathcal{D}er_{\mathbb{C}}(\log D) = \tilde{\Theta}_h \oplus \mathcal{O}_X \delta_d$. Let us call

$$\tilde{\Phi} : \text{Sym } \mathcal{D}er_{\mathbb{C}}(\mathcal{O}_X) = \text{gr } \mathcal{D}_X = \mathcal{O}_X[\xi] \rightarrow \text{Rees}(\mathcal{J}_D), \quad \tilde{\Phi}(\xi_j) = h'_{x_j} t,$$

the induced graded map, which is surjective.

Let us consider the augmented Koszul complex over $\text{gr } \mathcal{D}_X = \mathcal{O}_X[\xi]$ associated with $\tilde{\Theta}_h \equiv \sigma(\tilde{\Theta}_h) \subset \text{gr}^1 \mathcal{D}_X$

$$\begin{aligned} \mathbf{K}^\bullet &:= 0 \rightarrow \text{gr } \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^{d-1} \tilde{\Theta}_h \xrightarrow{d_{-d+1}} \dots \xrightarrow{d_{-2}} \text{gr } \mathcal{D}_X \otimes_{\mathcal{O}_X} \bigwedge^1 \tilde{\Theta}_h \xrightarrow{d_{-1}} \\ &\xrightarrow{d_{-1}} \text{gr } \mathcal{D}_X \xrightarrow{d_0} \text{Rees}(\mathcal{J}_D) \rightarrow 0, \\ d_{-k}(F \otimes (\sigma_1 \wedge \dots \wedge \sigma_k)) &= \sum_{i=1}^k (-1)^{i-1} F \sigma_i \otimes (\sigma_1 \wedge \dots \wedge \hat{\sigma}_i \wedge \dots \wedge \sigma_k) \end{aligned}$$

with augmentation $d_0 = \tilde{\Phi}$. Since D is Koszul, $\sigma(\delta_1), \dots, \sigma(\delta_{d-1})$ is a regular sequence and so \mathbf{K}^\bullet is exact in degrees ≤ -1 . It is also exact in degrees ≥ 1 because $\tilde{\Phi}$ is surjective.

In order to prove that D is of linear Jacobian type we need to prove that \mathbf{K}^\bullet is exact in degree 0, but for that it is enough to prove the exactness at any point $q \in X - \{0\}$ (see [4, Proposition 5.4]), or in fact at any point $q \in D - \{0\}$ since \mathbf{K}^\bullet is exact outside D .

Let $q \in D - \{0\}$ be a point. From the Koszul hypothesis we know (cf. [4, Corollary 1.9]) that the zero locus of the symbols $\sigma(\delta_i) = \sum_{j=1}^d a_{ij} \xi_j$, $i = 1, \dots, d$, in the cotangent bundle T^*X has dimension d , and so the zero locus of the coefficients a_{ij} , $i, j = 1, \dots, d$, is the origin and there is a logarithmic derivation with respect to D non vanishing at q . We can integrate it and deduce that (X, D, p) is analytically isomorphic to $(\mathbb{C}^{d-1}, D', 0) \times (\mathbb{C}, 0)$, where $(D', 0) \subset (\mathbb{C}^{d-1}, 0)$ is a germ of hypersurface. It is easy to see that $(D', 0)$ is again a germ of Koszul free divisor (cf. [4, Proposition 1.10]) and strongly Euler homogeneous (cf. Remark (1.5)). By the induction hypothesis, we deduce that $(D', 0)$ is of linear Jacobian type. So D is of linear Jacobian type at q and \mathbf{K}^\bullet is exact in degree 0 at q .

Equivalences (b) \Leftrightarrow (c), (c) \Leftrightarrow (d) have been proven respectively in Proposition (1.9) and Proposition (2.1).

(d) \Rightarrow (e) It is a consequence of Corollary (4.6) and Corollary (2.4).

(e) \Rightarrow (a) We proceed by induction on $\dim X$. If $\dim X = 2$ we know from [3] that D is locally quasi-homogeneous and so it is strongly Euler homogeneous. Assume that the implication (c) \Rightarrow (a) is true whenever the dimension of the ambient manifold is $d - 1$ and assume now that $\dim X = d \geq 3$. From [24, Corollary 1.8] and [5, Corollary 4.3] or [10, Criterion 4.1], we deduce that D is Euler homogeneous. It remains to prove that for any point $p \in D$ there is an Euler vector field with respect to some (and hence, for any) reduced equation of (D, p) , vanishing on p . Let us take a such reduced equation $h = 0$ and an Euler vector field χ on a neighborhood of p such that $\chi(h) = h$. If χ vanishes at p , we are done. If not, we can integrate χ and deduce that (X, D, p) is analytically isomorphic to $(\mathbb{C}^{d-1}, D', 0) \times (\mathbb{C}, 0)$, where $(D', 0) \subset (\mathbb{C}^{d-1}, 0)$ is a germ of hypersurface. We deduce as before that $(D', 0)$ is again a germ of Koszul free divisor. Since D satisfies the logarithmic comparison theorem, we deduce easily from [5, Corollary 4.2] or [10, Criterion 4.1] that D' also does. By the induction hypothesis, D' is strongly Euler homogeneous and so D is also strongly Euler homogeneous. Q.E.D.

5 The case of logarithmic connections

In this section we generalize the preceding results to the case of integrable logarithmic connections. Proofs remain essentially the same and will be only sketched. This generalization is based on [7, Theorem 3.1, Corollary 3.2], and so we will assume that our germ of free divisor $(D, 0) \subset (\mathbb{C}^d, 0)$, with reduced equation $h : (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}, 0)$, is of linear Jacobian type. We keep the notations of §2 and §3.

Let \mathcal{E} be a germ at 0 of integrable logarithmic connection with respect D , i.e. \mathcal{E} is a free \mathcal{O} -module of finite rank endowed with a left \mathcal{V} -module structure. As explained in [7, § 3.1], we consider the logarithmic Bernstein-Kashiwara module $\mathcal{E}[s]h^s$ inside the meromorphic connection $\mathcal{E}[s, h^{-1}]h^s$. Corollary 3.2 in *loc. cit.* tells us that the canonical map

$$\mathcal{D}_X[s] \otimes_{\mathcal{V}_X[s]}^{\mathbf{L}} \mathcal{E}[s]h^s \rightarrow \mathcal{D}_X[s]\mathcal{E}[s]h^s$$

is an isomorphism in $D_f^b(\mathcal{D}[s])$.

More generally, for any $q(s) \in \mathbb{C}[s]$ we consider the $q(s)$ -Bernstein module associated with \mathcal{E}

$$\mathcal{E}[s]h^{q(s)} := \mathcal{E}[s] \otimes_{\mathcal{O}[s]} \mathcal{O}[s]h^{q(s)} \subset \mathcal{E}[s, h^{-1}]h^{q(s)} := \mathcal{E}[s, h^{-1}] \otimes_{\mathcal{O}[s]} \mathcal{O}[s, h^{-1}]h^{q(s)}$$

as in Definition (2.5). In the same way as in Lemma (2.6) and Proposition (2.7) we prove that for any automorphism of \mathbb{C} -algebras $\varphi : \mathbb{C}[s] \rightarrow \mathbb{C}[s]$ the canonical map

$$\varphi^* \left(\mathcal{D}[s]\mathcal{E}[s]h^{q(s)} \right) := \mathcal{D}[s] \otimes_{\varphi} (\mathcal{D}[s]\mathcal{E}[s]h^s) \rightarrow \mathcal{D}[s]\mathcal{E}[s]h^{\varphi(q(s))} \quad (2)$$

induced by $\overline{\varphi} : \mathcal{O}[s, h^{-1}]h^{q(s)} \rightarrow \mathcal{O}[s, h^{-1}]h^{\varphi(q(s))}$ is an isomorphism of left $\mathcal{D}[s]$ -modules. Also, the canonical map

$$\mathcal{D}[s] \overset{\mathbf{L}}{\otimes}_{\mathcal{T}[s]} \left(\mathcal{E}[s]h^{\varphi(s)} \right) \rightarrow \mathcal{D}[s]\mathcal{E}[s]h^{\varphi(s)}$$

is an isomorphism in the derived category of left $\mathcal{D}[s]$ -modules. As in corollary (3.7) we obtain a canonical isomorphism

$$\mathbb{D} \left(\mathcal{D}[s]\mathcal{E}[s]h^{\varphi(s)} \right) \simeq \mathcal{D}[s]\mathcal{E}^*[s]h^{-\varphi(s)-1}.$$

Recall that the Bernstein-Sato polynomial of \mathcal{E} is defined as the minimal polynomial of the action of s on the quotient $\mathcal{Q}_{\mathcal{E}} := \mathcal{D}[s]\mathcal{E}[s]h^s / \mathcal{D}[s]\mathcal{E}[s]h^{s+1}$ and it is denoted by $b_{\mathcal{E}}(s)$ ([7, Remark 3.5]). The existence of a non-zero $b_{\mathcal{E}}(s)$ is a straightforward consequence of the existence of non trivial Bernstein-Sato functional equations with respect to sections of holonomic \mathcal{D} -modules ([16, Theorem 2.7]; see also [17]).

Now we are ready to state and prove the announced extension of Theorem (4.1) to the case of arbitrary logarithmic connections.

(5.1) THEOREM. *Let $(D, 0) \subset (\mathbb{C}^d, 0)$ be a germ of free divisor of linear Jacobian type with reduced equation $h : (\mathbb{C}^d, 0) \rightarrow (\mathbb{C}, 0)$ and \mathcal{E} a germ at 0 of integrable logarithmic connection with respect to D . Then the Bernstein-Sato polynomials of \mathcal{E} and of its dual \mathcal{E}^* are related by the equality*

$$b_{\mathcal{E}}(s) = \pm b_{\mathcal{E}^*}(-s-2).$$

PROOF. We proceed as in the proof of Theorem (4.1). Let us consider the exact sequence of left $\mathcal{D}[s]$ -modules

$$0 \rightarrow \mathcal{D}[s]\mathcal{E}[s]h^{s+1} \rightarrow \mathcal{D}[s]\mathcal{E}[s]h^s \rightarrow \mathcal{Q}_{\mathcal{E}} \rightarrow 0.$$

By applying the duality functor \mathbb{D} we obtain a triangle

$$\mathbb{D}(\mathcal{Q}_{\mathcal{E}}) \rightarrow \mathbb{D}(\mathcal{D}[s]\mathcal{E}[s]h^s) \rightarrow \mathbb{D}(\mathcal{D}[s]\mathcal{E}[s]h^{s+1}) \overset{\pm 1}{\rightarrow}$$

in which the second arrow corresponds to the inclusion $\mathcal{D}[s]\mathcal{E}^*[s]h^{-s-1} \rightarrow \mathcal{D}[s]\mathcal{E}^*[s]h^{-s-2}$, $\mathbb{D}(\mathcal{Q}_{\mathcal{E}})$ is concentrated in degree 1 and there is an exact sequence of left $\mathcal{D}[s]$ -modules

$$0 \rightarrow \mathcal{D}[s]\mathcal{E}^*[s]h^{-s-1} \rightarrow \mathcal{D}[s]\mathcal{E}^*[s]h^{-s-2} \rightarrow \mathbb{D}^1(\mathcal{Q}_{\mathcal{E}}) \rightarrow 0.$$

Let $\varphi : \mathbb{C}[s] \rightarrow \mathbb{C}[s]$ be the automorphism of \mathbb{C} -algebras determined by $\varphi(s) = -s-2$ and let us consider the exact sequence of left $\mathcal{D}[s]$ -modules

$$0 \rightarrow \mathcal{D}[s]\mathcal{E}^*[s]h^{s+1} \rightarrow \mathcal{D}[s]\mathcal{E}^*[s]h^s \rightarrow \mathcal{Q}_{\mathcal{E}^*} \rightarrow 0.$$

From (2) we deduce an isomorphism $\varphi^*(\mathcal{Q}_{\mathcal{E}^*}) \simeq \mathbb{D}^1(\mathcal{Q}_{\mathcal{E}})$ and so the minimal polynomial of the action of s on $\mathbb{D}^1(\mathcal{Q}_{\mathcal{E}})$ is $\varphi(b_{\mathcal{E}^*}(s)) = b_{\mathcal{E}^*}(-s-2)$. On the other hand, the action of s on $\mathbb{D}^1(\mathcal{Q}_{\mathcal{E}})$ is annihilated by $b_{\mathcal{E}}(s)$ and we conclude that $b_{\mathcal{E}}(s)$ is a multiple of $b_{\mathcal{E}^*}(-s-2)$.

In a symmetric way we deduce that $b_{\mathcal{E}^*}(s)$ is a multiple of $b_{\mathcal{E}}(-s-2)$, or equivalently $b_{\mathcal{E}^*}(-s-2)$ is a multiple of $b_{\mathcal{E}}(s)$, and so $b_{\mathcal{E}}(s) = \pm b_{\mathcal{E}^*}(-s-2)$. Q.E.D.

References

- [1] J. Bernstein. The analytic continuation of generalized functions with respect to a parameter. *Funct. Anal. Appl.*, 6 (1972), 26–40.
- [2] F. J. Calderón-Moreno. Logarithmic differential operators and logarithmic de Rham complexes relative to a free divisor. *Ann. Sci. École Norm. Sup. (4)*, 32(5) (1999), 701–714. (arXiv:math/9807047).
- [3] F. J. Calderón-Moreno, D. Q. Mond, L. Narváez-Macarro and F. J. Castro-Jiménez. Logarithmic Cohomology of the Complement of a Plane Curve. *Comment. Math. Helv.*, 77(1) (2002), 24–38. (arXiv:math/9807047).
- [4] F. J. Calderón-Moreno and L. Narváez-Macarro. The module $\mathcal{D}f^s$ for locally quasi-homogeneous free divisors. *Compositio Math.*, 134(1) (2002), 59–74. (arXiv:math/0206262).
- [5] F. J. Calderón Moreno and L. Narváez Macarro. Dualité et comparaison sur les complexes de de Rham logarithmiques par rapport aux diviseurs libres. *Ann. Inst. Fourier (Grenoble)*, 55(1) (2005), 47–75. (arXiv:math/0411045).
- [6] F. J. Calderón Moreno and L. Narváez Macarro. A mixed associativity formula for tensor products over two Lie-Rinehart algebras. *Ann. Univ. Ferrara - Sez. VII - Sc. Mat.* Vol. LI (2005), 105–118.
- [7] F. J. Calderón Moreno and L. Narváez Macarro. On the logarithmic comparison theorem for integrable logarithmic connections. *Proc. London Math. Soc.*, (3) 98 (2009), 585–606. (arXiv:math/0603003).
- [8] F. J. Castro-Jiménez, D. Mond, and L. Narváez-Macarro. Cohomology of the complement of a free divisor. *Trans. Amer. Math. Soc.*, 348 (1996), 3037–3049.
- [9] F. J. Castro-Jiménez and J. M. Ucha-Enríquez. Free divisors and duality for \mathcal{D} -modules. *Proc. Steklov Inst. Math.*, 238 (2002), 88–96. (arXiv:math/0103085).
- [10] F. J. Castro-Jiménez and J. M. Ucha-Enríquez. Testing the logarithmic comparison theorem for free divisors. *Experiment. Math.*, 13(4) (2004), 441–449.
- [11] S. Chemla. A duality property for complex Lie algebroids. *Math. Z.*, 232(2) (1999), 367–388.
- [12] H. Esnault and E. Viehweg. Logarithmic De Rham complexes and vanishing theorems. *Invent. Math.*, 86 (1986), 161–194.
- [13] M. Granger, D. Mond, A. Nieto-Reyes and M. Schulze. Linear free divisors and the global logarithmic comparison theorem. *Ann. Inst. Fourier (Grenoble)*, 59(2) (2009), 811–850. (arXiv:math/0607045).
- [14] M. Granger and M. Schulze. On the Symmetry of b-Functions of Linear Free Divisors. *Publ. RIMS Kyoto Univ.*, 46 (2010), 479–506. (arXiv:0807.0560).

- [15] M. Kashiwara. B-Functions and Holonomic Systems. *Invent. Math.* 38 (1976), 33–53.
- [16] M. Kashiwara. On the Holonomic Systems of Linear Differential Equations, II. *Invent. Math.* 49 (1978), 121–135.
- [17] Z. Mebkhout and L. Narváez-Macarro. La théorie du polynôme de Bernstein-Sato pour les algèbres de Tate et de Dwork-Monsky-Washnitzer. *Ann. Sci. École Norm. Sup. (4)*, 24(2) (1991), 227–256.
- [18] H. Nakayama and J. Sekiguchi. Determination of b -functions of polynomials defining Saito Free Divisors related with simple curve singularities of types E_6, E_7, E_8 . *Kumamoto Math. J.*, 22 (2009), 1–15.
- [19] L. Narváez Macarro. Linearity conditions on the Jacobian ideal and logarithmic-meromorphic comparison for free divisors. In “Singularities I, Algebraic and Analytic Aspects (International Conference in Honor of the 60th Birthday of Lê Dũng Tráng, January 8–26, 2007, Cuernavaca, México)”, 245–269. Contemporary Mathematics, 474, AMS, 2008. (arXiv:0804.2219).
- [20] G. S. Rinehart. Differential forms on general commutative algebras. *Trans. Amer. Math. Soc.*, 108 (1963), 195–222.
- [21] K. Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo*, 27 (1980), 265–291.
- [22] C. Sevenheck. Bernstein polynomials and spectral numbers for linear free divisors. *Ann. Inst. Fourier (Grenoble)*, 61(1) (2011), 379–400. (arXiv:0905.0971).
- [23] B. Teissier. Cycles évanescents, sections planes et conditions de Whitney. In *Singularités à Cargèse (Rencontre Singularités Géom. Anal., Inst. Études Sci., Cargèse, 1972)*, pages 285–362. Astérisque, Nos. 7 et 8. Soc. Math. France, Paris, 1973.
- [24] T. Torrelli. On meromorphic functions defined by a differential system of order 1. *Bull. Soc. Math. France*, 132 (2004), 591–612.
- [25] J. M. Ucha Enríquez. Métodos constructivos en álgebras de operadores diferenciales. Univ. Sevilla, September 1999. Ph.D.
- [26] W. V. Vasconcelos. *Computational methods in commutative algebra and algebraic geometry, Algorithms and Computation in Mathematics*, 2. Springer Verlag, New York, 1998.
- [27] T. Yano. On the theory of b -functions. *Publ. RIMS Kyoto Univ.*, 14 (1978), 111–202.

Departamento de Álgebra & Instituto de Matemáticas (IMUS)
 Facultad de Matemáticas, Universidad de Sevilla,
 Apdo. 1160, 41080 Sevilla, Spain.
E-mail: narvaez@algebra.us.es